Name:______________________________

MA 8663 Section 01 | *Practice Exam 1* | *February 18, 2020*

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work, answer or there is no justification to a solution you will receive little or no credit!

1. Let A be a operator defined on some separable Hilbert space $\mathcal{H}$ with dense domain $\mathcal{D}(A)$. Define the following quadratic form:

$$q_A(f) = \langle f, Af \rangle$$

for $f \in \mathcal{D}(A)$. Prove that A is symmetric if and only if the above quadratic form is real-valued.

2. Let A be a densely defined symmetric operator on some separable Hilbert space $\mathcal{H}$. Suppose that A has an orthonormal basis of eigenfunctions $\{\varphi_k\}$. Prove that A is essentially self-adjoint.

3. Define the following operator on $L^2[0, 2\pi]$:

$$A = -\frac{d^2}{dx^2}$$

with domain

$$\mathcal{D}(A) = \left\{ f \in C^2(0, 2\pi) : f(0) = f(2\pi), f'(0) = f'(2\pi) \right\} .$$

Prove that A is self-adjoint and show that $\sigma(A) = \{n^2 : n \in \mathbb{Z}\}$. Moreover compute the corresponding eigenspaces, the spectral projections and compute the spectral decomposition. In addition, let $B = \sqrt{A}$, prove that $\sigma(B) = \mathbb{N}$.

4. Let X be a Banach space, possibly a Hilbert space. Give an example of an unbounded injective operator $A : \mathrm{dom}(A) \to X$ with closed graph such that

$$0 \in \sigma(A) \cap \sigma(A^{-1}) .$$

Be sure to justify.

5. Let $a \in \mathbb{R}$ and define U_a as a translation by a on $\mathscr{S}(\mathbb{R})$ and $\mathscr{S}'(\mathbb{R})$, that is

$$(U_a f)(x) = f(x + a)$$

for all f in either space. Prove that

$$\frac{1}{a}(U_a - I) \to \frac{d}{dx}$$

as $a \to 0$ where the convergence is pointwise in the weak topology, i.e. in the $\mathscr{S}'(\mathbb{R})$ topology.

6. Let U be a bounded open subset of $\mathbb{R}^n$ with a C^1 boundary. Let $u \in W^{k,p}(U)$. Prove that if $k < \dfrac{n}{p}$, then $u \in L^q(U)$, where $\dfrac{1}{q} = \dfrac{1}{p} - \dfrac{k}{n}$. Moreover, show that

$$\|u\|_{L^q(U)} \leq C\|u\|_{W^{k,p}(U)}$$

7. Prove that if $u \in W^{1,p}(0,1)$ for some $1 \leq p < \infty$, then $u = \tilde{u}$ a.e., where $\tilde{u} \in AC(0,1)$. Moreover show that $\tilde{u}' \in L^p(0,1)$.

8. Let $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be the Fourier transform. That is:

$$\mathcal{F}(f) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\langle x,\lambda\rangle} f(x)\ dx$$

Prove that $\mathcal{F}$ is a unitary operator on $L^2(\mathbb{R}^n)$.